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Some properties of the integrable noncommutative sine-Gordon system

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Abstract: In this paper we continue the program, initiated in ref. [1], to investigate an integrable noncommutative version of the sine-Gordon model. We discuss the origin of the extra constraint which the field function has to satisfy in order to guarantee classical integrability. We show that the system of constraint plus dynamical equation of motion can be obtained by a suitable reduction of a noncommutative version of 4d self-dual Yang-Mills theory. The field equations can be derived from an action which is the sum of two WZNW actions with cosine potentials corresponding to a complexified noncommutative U(1) gauge group. A brief discussion of the relation with the bosonized noncommutative Thirring model is given. In spite of integrability we show that the S-matrix is acausal and particle production takes place.

Keywords: Field Theories in Lower Dimensions, Non-Commutative Geometry, Integrable Field Theories.

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1. Introduction

Field theories defined in a noncommutative (NC) geometry have been recently the subject of detailed investigations, partly because noncommutativity emerges naturally in the low-energy dynamics of branes in a constant B field [2], but also because they have interesting features of their own.

In this context, an interesting question is how noncommutativity could affect the dynamics of exactly solvable field theories, as for instance two-dimensional integrable theories. A common feature of these systems is that the existence of an infinite chain of *local* conserved currents is guaranteed by the fact that the equations of motion can be written as zero curvature conditions for a suitable set of covariant derivatives [3]. In some cases, as for example the ordinary sine-Gordon or sigma models, an action is also known which generates the integrable equations according to an action principle.

Noncommutative versions of these models are intuitively defined as models which reduce to the ordinary ones when the noncommutation parameter θ is removed. In general these generalizations are not unique as one can construct different NC equations of motion which collapse to the same expression when θ goes to zero.[1]

For two dimensional integrable systems, a general criterion to restrict the number of possible NC versions is to require the classical integrability to survive in NC geometry. This suggests that any NC generalization should be performed at the level of equations of motion by promoting the standard zero curvature techniques. This program has been worked out for a number of known integrable equations in refs. [4, 5].

[1]As a simple example, we can consider a scalar free field theory where the ordinary equations of motion can have different NC counterparts, as for instance the equation itself or $\partial_\mu(e_*^{-i\phi} * \partial^\mu e_*^{i\phi}) = 0$.

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In the case of the sine-Gordon system

$$S = \frac{1}{\pi\lambda^2}\int d^2z\,\left[\partial\phi\bar{\partial}\phi - 2\gamma(\cos\phi - 1)\right] \tag{1.1}$$

we might define the "natural" NC generalization as the model obtained by promoting the ordinary product to the $*$-product in the action (1.1) and consequently in the equations of motion which become $\partial\bar{\partial}\phi = \gamma\sin_*\phi$. However, this model seems to be affected by some problems both at the classical and the quantum level.

At the classical level it does not seem to be integrable since the ordinary currents promoted to NC currents by replacing the products with $*$-products are not conserved [1]. Moreover, we don't know how to find a systematic procedure to construct conserved currents since the equations of motion cannot be obtained as zero curvature conditions (a discussion about the lack of integrability for this system is also given in ref. [6]).

At the quantum level the renormalizability properties of the ordinary model (1.1) defined for $\lambda^2 < 4$ seem to be destroyed by NC. The reason is quite simple and can be understood by analyzing the structure of the divergences of the NC model compared to the ordinary ones [7, 8]. In the $\lambda^2 < 4$ regime the only divergences come from multitadpole diagrams. In the ordinary case the n-loop diagram gives a contribution $(\log m^2a^2)^n$ where a and m are the UV and IR cut-offs respectively. This result is independent of the number k of external fields and of the external momenta. As a consequence the total contribution at this order can be resummed as $\gamma(\log m^2a^2)^n(\cos\phi - 1)$ and the divergence is cancelled by renormalizing the coupling γ. This holds at any order n and the model is renormalizable.

In the NC case the generic vertex from the expansion of $\cos_*\phi$ brings nontrivial phase factors which depend on the momenta coming out of the vertex and on the NC parameter. The final configuration of phase factors associated to a given diagram depends on the order we use to contract the fields in the vertex. Therefore, in the NC case the ordinary n-loop diagram splits into a planar and a certain number of nonplanar configurations, where the planar one has a trivial phase factor whereas the nonplanar diagrams differ by the configuration of the phases (for a general discussion see refs. [9, 10]). The most general NC multitadpole diagram is built up by combining planar parts with nonplanar ones where two or more tadpoles are intertwined among themselves or with external legs. Since the nonplanar subdiagrams are convergent [10, 11] a generic n-loop diagram contributes to the divergences of the theory only if it contains a nontrivial planar subdiagram. However, different n-loop diagrams with different configurations of planar and nonplanar parts give divergent contributions whose coefficients depend on the number k of external fields and on the external momenta. A resummation of the divergences to produce a cosine potential is not possible anymore and the renormalization of the couplings of the model is not sufficient to make the theory finite at any order. Noncommutativity seems to deform the cosine potential at the quantum level and the theory loses the renormalizability properties of the corresponding commutative model.

Thus, the "natural" generalization of sine-Gordon is not satisfactory and one must look for a different NC generalization compatible with integrability and/or renormalizability.

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In ref. [1] we considered the problem of classical integrability. The method of the *bicomplex* implemented in NC geometry was used to construct a NC version of the sine-Gordon equation which is integrable, and the first few nontrivial currents were constructed and studied. The equations of motion for that system are quite unexpected and do not resemble the ones of the "natural" generalization. Precisely, what emerges in the NC case is a system of two coupled equations of motion (we use euclidean signature and complex coordinates $z = \frac{1}{\sqrt{2}}(x^0 + ix^1)$, $\bar{z} = \frac{1}{\sqrt{2}}(x^0 - ix^1)$)

$$\begin{aligned} 2i\bar{\partial}b &\equiv \bar{\partial}\left(e_*^{-\frac{i}{2}\phi} * \partial e_*^{\frac{i}{2}\phi} - e_*^{\frac{i}{2}\phi} * \partial e_*^{-\frac{i}{2}\phi}\right) = i\gamma \sin_* \phi \\ 2\bar{\partial}a &\equiv \bar{\partial}\left(e_*^{\frac{i}{2}\phi} * \partial e_*^{-\frac{i}{2}\phi} + e_*^{-\frac{i}{2}\phi} * \partial e_*^{\frac{i}{2}\phi}\right) = 0\,, \end{aligned} \tag{1.2}$$

where $f * g = f \exp(\frac{\theta}{2}(\overleftarrow{\partial}\overrightarrow{\bar{\partial}} - \overleftarrow{\bar{\partial}}\overrightarrow{\partial}))g$.

The first equation contains the potential term which is the "natural" generalization of the ordinary sine potential, whereas the other one has the structure of a conservation law and can be seen as imposing an extra condition on the system. In the commutative limit, the first equation reduces to the ordinary sine-Gordon equation, whereas the second one becomes trivial. The equations are in general complex and possess the Z_2 symmetry of the ordinary sine-Gordon (invariance under $\phi \to -\phi$).

The reason why integrability seems to require two equations of motions can be traced back to the general structure of unitary groups in NC geometry. In the bicomplex approach the ordinary equations are obtained as zero curvature conditions for covariant derivatives defined in terms of SU(2) gauge connections. If the same procedure is to be implemented in the noncommutative case, the group SU(2), which is known to be not closed in noncommutative geometry, has to be extended to a noncommutative U(2) group and a NC U(1) factor enters necessarily into the game. The appearance of the second equation in (1.2) for our NC integrable version of sine-Gordon is then a consequence of the fact that the fields develop a nontrivial trace part. We note that the pattern of equations we have found seems to be quite general and unavoidable if integrability is of concern. In fact, the same has been found in ref. [6] where a different but equivalent set of equations was proposed.

The presence of two equations of motion is in principle very restrictive and one may wonder whether the class of solutions is empty. To show that this is not the case, in ref. [1] solitonic solutions were constructed perturbatively which reduce to the ordinary solitons when we take the commutative limit. More generally, we observe that the second equation in (1.2) is automatically satisfied by any chiral or antichiral function. Therefore, we expect the class of solitonic solutions to be at least as large as the ordinary one. In the general case, instead, we expect the class of dynamical solutions to be smaller than the ordinary one because of the presence of the nontrivial constraint. However, since the constraint equation is one order higher with respect to the dynamical equation, order by order in the θ-expansion a solution always exists. This means that a Seiberg-Witten map between the NC and the ordinary model does not exist as a mapping between physical configurations, but it might be constructed as a mapping between equations of motion or conserved currents.

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The question which was left open in ref. [1] was the existence of an action for the set of equations (1.2). In this paper we give an action and discuss the relation of our model with the NC selfdual Yang-Mills theory and the NC Thirring model. Moreover, we discuss some properties of the corresponding S-matrix which, in spite of integrability, turns out to be acausal and not factorized.

2. Connection with NC selfdual Yang-Mills

The (anti-)selfdual Yang-Mills equation is well-known to describe a completely integrable classical system in four dimensions [12]. In the ordinary case the equations of motion for many two dimensional integrable systems, including sine-Gordon, can be obtained through dimensional reduction of the (A)SDYM equations [13].

A convenient description of the (A)SDYM system is the so called J-formulation, given in terms of a $SL(N,C)$ matrix-valued J field satisfying

$$\partial_{\bar{y}}\left(J^{-1}\partial_y J\right)+\partial_{\bar{z}}\left(J^{-1}\partial_z J\right)=0\,, \tag{2.1}$$

where y, $\bar{y}$, z, $\bar{z}$ are complex variables treated as formally independent.

In the ordinary case, the sine-Gordon equation can be obtained from (2.1) by taking J in $SL(2,C)$ to be [14]

$$J=J(u,z,\bar{z})=e^{\frac{z}{2}\sigma_i}e^{\frac{i}{2}u\sigma_j}e^{-\frac{\bar{z}}{2}\sigma_i}\,, \tag{2.2}$$

where $u=u(y,\bar{y})$ depends on y and $\bar{y}$ only and σ_i are the Pauli matrices.

A noncommutative version of the (anti-)selfdual Yang-Mills system can be naturally obtained [15] by promoting the variables y, $\bar{y}$, z and $\bar{z}$ to be noncommutative thus extending the ordinary products in (2.1) to $*$-products. In this case the J field lives in $\mathrm{GL}(N,C)$.

It has been shown [16] that NC SDYM naturally emerges from open $N=2$ strings in a B-field background. Moreover, in refs. [15, 17, 5] examples of reductions to two-dimensional NC systems were given. It was also argued that the NC deformation should preserve the integrability of the systems [18].

We now show that our NC version of the sine-Gordon equations can be derived through dimensional reduction from the NC SDYM equations. For this purpose we consider the NC version of equations (2.1) and choose J_* in $\mathrm{GL}(2,C)$ as

$$J_*=J_*(u,z,\bar{z})=e_*^{\frac{z}{2}\sigma_i}*e_*^{\frac{i}{2}u\sigma_j}*e_*^{-\frac{\bar{z}}{2}\sigma_i}\,. \tag{2.3}$$

This leads to the matrix equation

$$\partial_{\bar{y}}aI+i\left(\partial_{\bar{y}}b+\frac{1}{2}\sin_* u\right)\sigma_j=0\,, \tag{2.4}$$

where a and b have been defined in (1.2). Now, taking the trace we obtain $\partial_{\bar{y}}a=0$ which is the constraint equation in (1.2). As a consequence, the term proportional to σ_j gives rise to the dynamical equation in (1.2) for the particular choice $\gamma=-1$. Therefore we have shown that the equations of motion of the NC version of sine-Gordon proposed in ref. [1]

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can be obtained from a suitable reduction of the NC SDYM system as in the ordinary case. From this derivation the origin of the constraint appears even more clearly: it arises from setting to zero the trace part which the matrices in $\mathrm{GL}(2,C)$ naturally develop under $*$-multiplication.

Solving (2.4) for the particular choice $\sigma_j = \sigma_3$ we obtain the alternative set of equations

$$\begin{aligned}\bar\partial\left(e_*^{-\frac{i}{2}\phi} * \partial e_*^{\frac{i}{2}\phi}\right) &= \frac{i}{2}\gamma \sin_* \phi \\ \bar\partial\left(e_*^{\frac{i}{2}\phi} * \partial e_*^{-\frac{i}{2}\phi}\right) &= -\frac{i}{2}\gamma \sin_* \phi\,. \end{aligned} \tag{2.5}$$

Order by order in the θ-expansion the set of equations (1.2) and (2.5) are equivalent. Therefore, the set (2.5) is equally suitable for the description of an integrable noncommutative generalization of sine-Gordon.

Since our NC generalization of sine-Gordon is integrable, the present result gives support to the arguments in favor of the integrability of NC SDYM system.

We note that our equations of motion, Wick rotated to Minkowski, can also be obtained by suitable reduction of the $2+1$ integrable noncommutative model studied in ref. [19].

3. The action

We are now interested in the possibility of determining an action for the scalar field ϕ satisfying the system of eqs. (1.2). We are primarily motivated by the possibility to move on to a quantum description of the system.

In general, it is not easy to find an action for the dynamical equation (the first eq. in (1.2)) since ϕ is constrained by the second one. One possibility could be to implement the constraint by the use of a Lagrange multiplier.

We consider instead the equivalent set of equations (2.5). We rewrite them in the form

$$\begin{aligned}\bar\partial(g^{-1} * \partial g) &= \frac{1}{4}\gamma\left(g^2 - g^{-2}\right) \\ \bar\partial(g * \partial g^{-1}) &= -\frac{1}{4}\gamma\left(g^2 - g^{-2}\right), \end{aligned} \tag{3.1}$$

where we have defined $g \equiv e_*^{\frac{i}{2}\phi}$. Since ϕ is in general complex g can be seen as an element of a noncommutative complexified U(1). The gauge group valued function $\bar g \equiv (g^\dagger)^{-1} = e_*^{\frac{i}{2}\phi^\dagger}$ is subject to the equations

$$\begin{aligned}\bar\partial(\bar g * \partial \bar g^{-1}) &= -\frac{1}{4}\gamma\left(\bar g^2 - \bar g^{-2}\right) \\ \bar\partial(\bar g^{-1} * \partial \bar g) &= \frac{1}{4}\gamma\left(\bar g^2 - \bar g^{-2}\right) \end{aligned} \tag{3.2}$$

obtained by taking the h.c. of (3.1).

In order to determine the action it is convenient to concentrate on the first equation in (3.1) and the second one in (3.2) as the two independent complex equations of motion which describe the dynamics of our system.

We first note that the left-hand sides of equations (3.1) and (3.2) have the chiral structure which is well known to correspond to a NC version of the WZNW action [20]. Therefore we are led to consider the action

$$S[g,\bar{g}] = S[g] + S[\bar{g}]\,, \tag{3.3}$$

where, introducing the homotopy path $\hat{g}(t)$ such that $\hat{g}(0) = 1$, $\hat{g}(1) = g$ (t is a commuting parameter) we have defined

$$S[g] = \int d^2z \left[\partial g * \bar{\partial} g^{-1} + \int_0^1 dt\ \hat{g}^{-1} * \partial_t \hat{g} * [\hat{g}^{-1} * \partial \hat{g}, \hat{g}^{-1} * \bar{\partial}\hat{g}]_* - \frac{\gamma}{4}(g^2 + g^{-2} - 2)\right] \tag{3.4}$$

and similarly for $S[\bar{g}]$. The first part of the action can be recognized as the NC generalization of a complexified U(1) WZNW action [21].

To prove that this generates the correct equations, we should take the variation with respect to the ϕ field ($g = e_*^{\frac{i}{2}\phi}$) and deal with complications which follow from the fact that in the NC case the variation of an exponential is not proportional to the exponential itself. However, since the variation $\delta\phi$ is arbitrary, we can forget about its θ dependence and write $\frac{i}{2}\delta\phi = g^{-1}\delta g$, trading the variation with respect to ϕ with the variation with respect to g. Analogously, the variation with respect to $\phi^\dagger$ can be traded with the variation with respect to $\bar{g}$.

It is then a simple calculation to show that

$$\delta S[g] = \int d^2z\ 2g^{-1}\delta g \left[\bar{\partial}\left(g^{-1} * \partial g\right) - \frac{i}{2}\gamma \sin_* \phi\right] \tag{3.5}$$

from which we obtain the first equation in (3.1). Treating $\bar{g}$ as an independent variable an analogous derivation gives the second equation in (3.2) from $S[\bar{g}]$.

We note that, when ϕ is real, $g = \bar{g}$ and the action (3.3) reduces to $S_{\mathrm{real}}[g] = 2S_{WZW}[g] - \gamma(\cos_* \phi - 1)$. In general, since the two equations (1.2) are complex it would be inconsistent to restrict ourselves to real solutions. However, it is a matter of fact that the equations of motion become real when the field is real. Perturbatively in θ this can be proved order by order by direct inspection of the equations in ref. [1]. In particular, at a given order one can show that the imaginary part of the equations vanishes when the constraint and the equations of motion at lower orders are satisfied.

It would be interesting to obtain the action (3.3) from the dimensional reduction of the 4d SDYM action by generalizing to the NC case the procedure used in Ref. [14].

4. Connection with NC Thirring model

In the ordinary case the equivalence between Thirring and sine-Gordon models [7] can be proven at the level of functional integrals by implementing the bosonization prescription [22]–[24] on the fermions. The same procedure has been worked out in NC geometry [25, 26]. Starting from the NC version of Thirring described by

$$S_T = \int d^2x \left[\bar{\psi} i\gamma^\mu \partial_\mu \psi + m\bar{\psi}\psi - \frac{\lambda}{2}(\bar{\psi} * \gamma^\mu \psi)(\bar{\psi} * \gamma_\mu \psi)\right] \tag{4.1}$$

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the bosonization prescription gives rise to the action for the bosonized NC massive Thirring model which turns out to be a NC WZNW action supplemented by a cosine potential term for the NC U(1) group valued field which enters the bosonization of the fermionic currents. In particular, in the most recent paper in ref. [25] it has been shown that working in euclidean space the massless Thirring action corresponds to the sum of two WZNW actions once a suitable choice for the regularization parameter is made. Moreover, in ref. [26] it was proven that the bosonization of the mass term in (4.1) gives rise to a cosine potential for the scalar field with coupling constant proportional to m.

The main observation is that our action (3.3) is the sum of two NC WZNW actions plus cosine potential terms for the pair of $\mathrm{U}(1)_C$ group valued fields g and $\bar{g}$, considered as independent. Therefore, our action can be interpreted as coming from the bosonization of the massive NC Thirring model, in agreement with the results in refs. [25, 26].

We have shown that even in the NC case the sine-Gordon field can be interpreted as the scalar field which enters the bosonization of the Thirring model, so proving that the equivalence between Thirring and sine-Gordon can be maintained in NC generalizations of these models. Moreover, the classical integrability of our NC version of sine-Gordon proven in ref. [1] should automatically guarantee the integrability of the NC Thirring model.

In the particular case of zero coupling ($\gamma = 0$), the equations (3.3) and (2.5) correspond to the action and the equations of motion for a NC U(1) WZNW model [20], respectively. Again, we can use the results of ref. [1] to prove the classical integrability of the NC U(1) WZNW model and construct explicitly its conserved currents.

5. Properties of the S-matrix

It is well known that in integrable commutative field theories there is no particle production and the S-matrix factorizes. In the noncommutative case properties of the S-matrix have been investigated for two specific models: The $\lambda\Phi^4$ theory in two dimensions [27] and the nonintegrable "natural" NC generalization the the sine-Gordon model [6]. In the first reference a very pathological acausal behavior was observed due to the space and time noncommutativity. For an incoming wave packet the scattering produces an advanced wave which arrives at the origin before the incoming wave. In the second model investigated it was found that particle production occurs. The tree level $2 \to 4$ amplitude does not vanish.

It might be hoped that classical integrability would alleviate these pathologies. In the NC integrable sine-Gordon case, since we have an action, it is possible to investigate these issues. As described below we have computed the scattering amplitude for the $2 \to 2$ process and found that the acausality of ref. [27] is not cured by integrability. We have also computed the production amplitudes for the processes $2 \to 3$ and $2 \to 4$ and found that they don't vanish.

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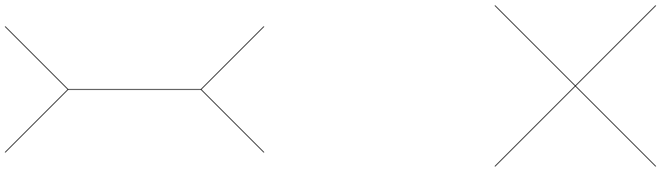

Figure 1: Tree level $2 \to 2$ amplitude.

We started from our action (3.4) rewritten in terms of Minkowski space coordinates x^0, x^1 and real fields ($g = e_*^{\frac{i}{2}\phi}$, $\hat{g}(t) = e_*^{\frac{i}{2}t\phi}$ with ϕ real)

$$
\begin{aligned}
S[g] = & -\frac{1}{2}\int d^2x\; g^{-1} * \partial^\mu g * g^{-1} * \partial_\mu g - \frac{1}{3}\int d^3x\; \epsilon^{\mu\nu\rho}\hat{g}^{-1} * \partial_\mu \hat{g} * \hat{g}^{-1} * \partial_\nu \hat{g} * \hat{g}^{-1} * \partial_\rho \hat{g} + \\
& +\frac{\gamma}{4}\int d^2x (g^2 + g^{-2} - 2)\,,
\end{aligned}
\tag{5.1}
$$

where $f * g = f e^{\frac{i}{2}\theta\epsilon^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu} g$, and we derived the following Feynman's rules

- The propagator

$$
G(q) = \frac{4i}{q^2 - 2\gamma}\,. \tag{5.2}
$$

- The vertices

$$
\begin{aligned}
v_3(k_1,\dots,k_3) &= \frac{2}{2^3\cdot 3!}\epsilon^{\mu\nu}k_{1\mu}k_{2\nu}F(k_1,\dots,k_3)\\
v_4(k_1,\dots,k_4) &= i\left(-\frac{1}{2^4\cdot 4!}\left(k_1^2 + 3k_1\cdot k_3\right) + \frac{\gamma}{2\cdot 4!}\right)F(k_1,\dots,k_4)\\
v_5(k_1,\dots,k_5) &= -\frac{2\epsilon^{\mu\nu}}{2^5\cdot 5!}\left(k_{1\mu}k_{2\nu} - k_{1\mu}k_{3\nu} + 2k_{1\mu}k_{4\nu}\right)F(k_1,\dots,k_5)\\
v_6(k_1,\dots,k_6) &= i\left[\frac{1}{2^6\cdot 6!}\left(k_1^2 + 5k_1\cdot k_3 - 5k_1\cdot k_4 + 5k_1\cdot k_5\right) - \frac{\gamma}{2\cdot 6!}\right]F(k_1,\dots,k_6)\,,
\end{aligned}
\tag{5.3}
$$

where

$$
F(k_1,\dots,k_n) = \exp\left(-\frac{i}{2}\sum_{i<j} k_i \times k_j\right) \tag{5.4}
$$

is the phase factor coming from the $*$-products in the action (we have indicated $a\times b = \theta\epsilon^{\mu\nu}a_\mu b_\nu$), k_i are all incoming momenta and we used momentum conservation.

At tree level the $2\to 2$ process is described by the diagrams with the topologies in figure 1.

Including contributions from the various channels and using the three point and four point vertices of eqs. (5.3) we obtained for the scattering amplitude the expression

$$
-\frac{i}{2}E^2p^2\left(\frac{1}{2E^2-\gamma} - \frac{1}{2p^2+\gamma}\right)\sin^2(pE\theta) + i\frac{\gamma}{2}\cos^2(pE\theta)\,, \tag{5.5}
$$

where p is the center of mass momentum and $E = \sqrt{p^2 + 2\gamma}$.

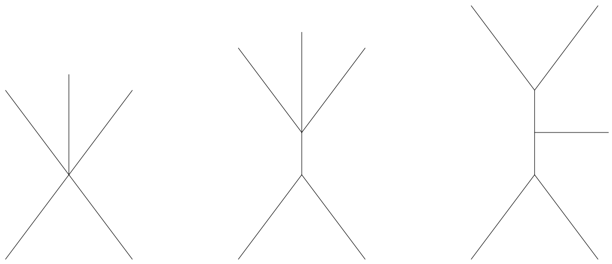

Figure 2: Tree level $2 \to 3$ amplitude.

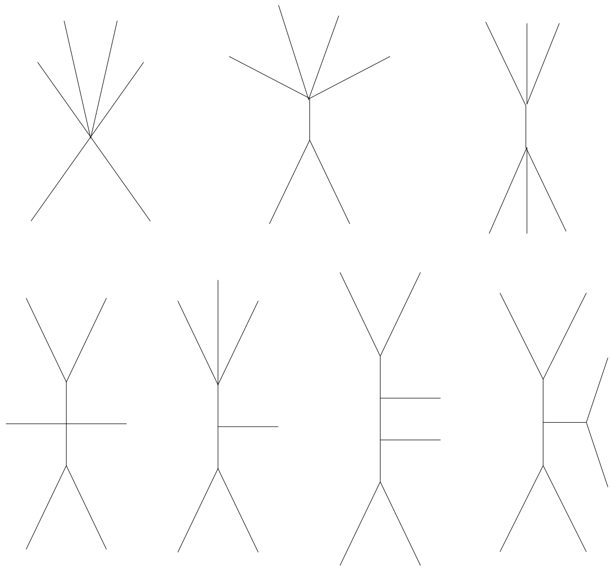

Figure 3: Tree level $2 \to 4$ amplitude.

For comparison with ref. [27] this should be multiplied by an incoming wave packet

$$\Phi_{in}(p) \sim \left(e^{-\frac{(p-p_0)^2}{\lambda}} + e^{-\frac{(p+p_0)^2}{\lambda}} \right) \tag{5.6}$$

and Fourier transformed with e^{ipx}. We have not attempted to carry out the Fourier transform integration. However, we note that for p_0 very large E and p are concentrated around large values and the scattering amplitude assumes the form

$$i\frac{\gamma}{4}\sin^2(pE\theta) + i\frac{\gamma}{2}\cos^2(pE\theta) \tag{5.7}$$

which is equivalent to the result in Ref. [27], leading to the same acausal pathology.[2]

We describe now the computation of the production amplitudes $2 \to 3$ and $2 \to 4$. At tree level the contributions are drawn in figures 2 and 3, respectively.

[2]It is somewhat tantalizing that a change in the relative coefficient between the two terms would lead to a removal of the trigonometric factors which are responsible for the acausal behavior.

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For any topology the different possible channels must be taken into account. This, as well as the complicated expressions for the vertices, has led us to use an algebraic manipulation program computer. We used *Mathematica*© to symmetrize completely the vertices (5.3). This allows to take automatically into account the different diagrams obtained by exchanging momenta entering a given vertex. The contribution from each diagram was obtained as a product of the combinatorial factor, the relevant vertices and propagators. Due to the length of the program it was impossible to handle the calculation in a complete analytic way. Instead, the program was run with assigned values of the momenta and arbitrary θ and γ. For both the $2 \to 3$ and $2 \to 4$ processes the result is nonvanishing. As a check of our calculation we mention that the production amplitudes vanish when we set $\theta = 0$, for any value of the coupling and the momenta.

This investigation has been carried on in the particular case $\phi = \phi^\dagger$. However, it can be easily proved that turning on a nontrivial imaginary part for the ϕ field does not cure the previous pathologies.

6. Conclusions

In this paper we have investigated some properties of the integrable NC sine-Gordon system proposed in ref. [1]. We succeeded in constructing an action which turned out to be a WZNW action for a noncommutative, complexified U(1) augmented by a cosine potential. We have shown that even in the NC case there is a duality relation between our integrable NC sine-Gordon model and the NC Thirring.

NC WZNW models have been shown to be one-loop renormalizable [28]. This suggests that the NC sine-Gordon model proposed in ref. [1] is not only integrable but it might lead to a well-defined quantized model, giving support to the existence of a possible relation between integrability and renormalizability.

Armed with our action we investigated some properties of the S-matrix for elementary excitations. However, in contradistinction to the commutative case, the S-matrix turned out to be acausal and nonfactorizable.[3] The reason for the acausal behavior has been discussed in ref. [27] where it was pointed out that noncommutativity induces a backward-in-time effect because of the presence of certain phase factors. It appears that in our case this effect is still present in spite of integrability.

It is not clear why the presence of an infinite number of local conserved currents (local in the sense that they are not expressed as integrals of certain densities) does not guarantee factorization and absence of production in the S-matrix as it does in the commutative case. The standard proofs of factorization use, among other assumptions, the mutual commutativity of the charges — a property we have not been able to check so far because of the complicated nature of the currents. But even if the charges were to commute the possibility of defining them as powers of momenta, as required in the proofs, could be spoiled by acausal effects which prevent a clear distinction between incoming and outgoing particles. Indeed, this may indicate some fundamental inconsistency of the model as a field

[3]Other problems of the S-matrix have been discussed in ref. [29].

theory describing scattering in 1+1 dimensional Minkowski space (simply going to euclidean signature does not change the factorizability properties of the S-matrix). However, it is conceivable that in euclidean space the model could be used to describe some statistical mechanics system and this possibility might be worth investigating.

In a series of papers [30] a different approach to quantum NC theories has been proposed when the time variable is not commuting. In particular, the way one computes Green's functions is different there, leading to a modified definition of the S-matrix. It would be interesting to redo our calculations in that approach to see whether a well-defined factorized S-matrix for our model can be constructed. In this context it would be also interesting to investigate the scattering of solitons present in our model [1]. To this end, since our model is a reduction of the $2+1$ integrable model studied in [31], it might be possible to exploit the results of those papers concerning multi-solitons and their scattering to investigate the same issues in our case.

The model we have proposed in the present paper describes the propagation of two interacting scalar fields $Re\phi$, $Im\phi$. The particular form of the interaction follows from the choice of the U(2) matrices made in [1] for the bicomplex formulation or, equivalently, from the particular ansatz (2.3) in the reduction from NC SDYM (in the commutative limit and for $Im\phi = 0$ we are back to the ordinary sine-Gordon). In the commutative case the ansatz (2.2) depends on a single real field and, independently of the choice of the Pauli matrices in the exponentials, we obtain the same equations of motion. In the NC case the lack of decoupling of the U(1) subgroup requires the introduction of two fields. This implies that in general we can make an ansatz $J_* = e_*^{\frac{\bar{z}}{2}\sigma_i} g(y,\bar{y}) e_*^{-\frac{\bar{z}}{2}\sigma_i}$ where g is a group valued field which depends on two scalar fields in such a way as to reduce to the ordinary ansatz (2.2) in the commutative limit and for a suitable identification of the two fields. In principle there are different choices for g as a function of the two fields satisfying this requirement. Different choices may be inequivalent and describe different but still integrable dynamics for the two fields. Therefore, an interesting question is whether an ansatz slightly different from (2.3) exists which would give rise to an integrable system described by a consistent, factorized S-matrix. We might expect that if such an ansatz exists it should introduce a *different* interaction between the two fields and this might cure the pathological behavior of the present scattering matrix. If such a possibility exists it would be interesting to compare the two different reductions to understand in the NC case what really drives the system to be integrable in the sense of having a well-defined, factorized S-matrix since the existence of an infinite number of local conservation laws does not appear to be sufficient. These issues will be discussed in a forthcoming paper [32].

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